

## Reciprocal Optimal Control Problems\*

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### 1. INTRODUCTION

Consider the standard optimal control problem of minimizing a given functional  $J$  subject to a differential equation for the motion of the controlled system and inequality constraints on the controlling variable. This is the *primal problem*. A problem of maximizing a new functional  $I$ , subject to certain constraints, such that  $\max I = \min J$ , is called a *reciprocal problem*. In addition, one requires that the functions minimizing  $J$  be identical to the functions maximizing  $I$ . This problem is termed here "reciprocal" rather than "dual" because the reciprocal of the reciprocal problem is not the primal, as is usually required of a duality relationship.

Interest in reciprocal problems lies primarily in the possibility of obtaining bounds for  $\min J$  of the primal problem in the course of a numerical successive approximation solution. By substituting these successive approximations in the reciprocal problem, or solving the reciprocal problem separately, one gets a sequence of lower bounds of  $\min J$  which helps to ascertain the convergence of the iterative procedure and to determine the stopping point. In addition, the reciprocal problem may be more tractable numerically than the primal one.

Here, a basic reciprocal problem and several variations of it are derived using inequalities based on convexity assumptions. Included as special cases are a reciprocal problem for linear systems derived by Hanson [1], and a reciprocal problem presented by Pearson [2]<sup>1</sup> and utilized for a linear system by Ringlee [3]. A reciprocal to a particular variational problem based on a convexity assumption was given by Bellman [4].

A different approach leading to essentially the same reciprocal problems, discovered by Trefftz (1927) and Friedrichs (1929), is described by

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<sup>1</sup> See also "Reciprocity and Duality in Control Programming Problems," by Pearson, this Journal, Vol. 10, No. 2, April 1965.

Courant [5] for a simple variational problem with no side constraints. The difficulty with this approach is the needed assumption of the existence of unique solutions in a neighborhood of the optimal one—an assumption which is very difficult to check.

The program of this paper is as follows. In Section 2 the standard primal optimal control problem is formulated and the relevant necessary conditions summarized. The convexity assumptions used to derive the reciprocal problems are stated in Section 3. Assumptions  $A_1$  and  $A_2$  are needed for all the reciprocal problems  $R_1$  through  $R_6$ , while the addition of  $A_3$  is needed for the reciprocal problem  $R_5$ , and the addition of  $A_4$  to  $A_1$ ,  $A_2$ , and  $A_3$  is needed for  $R_6$ . In Section 4 the basic reciprocal problem  $R_1$  is presented and  $R_2$  through  $R_6$  are derived from  $R_1$  by simply adding constraints and assumptions as is summarized in Fig. 1. The reciprocity theorem is stated and proved in Section 5 and the implications of the convexity assumptions are discussed in Section 6. Further comments on the relative merits and applicability of the reciprocal problems appear in the Conclusion.

## 2. THE PRIMAL PROBLEM

The problem formulation follows, to a large extent, that of Berkovitz [6]. As in his paper, the same symbol represents a vector as both a row and a column vector, obviating transposition of matrices. Thus,  $xy$  is the scalar product of the vectors  $x$  and  $y$ . If  $R(t, x, u)$  is an  $r$ -vector and  $x$  an  $n$ -vector, then  $R_x$  is the  $(r \times n)$ -matrix  $(\partial R^i / \partial x^j)$ ; if  $\mu$  is an  $r$ -vector then  $\mu R_x$  is an  $(1 \times n)$ -matrix. Superscripts denote the components of a vector and subscripts distinguish between vectors.

The system to be controlled is described by the vector differential equation

$$\dot{x} = f(t, x, u), \quad x(t_0) = x_0, \quad (1)$$

where  $x = (x^1, \dots, x^n)$  is the *state*,  $u = (u^1, \dots, u^m)$  is the *control* and  $t$  is the time. A bounded and piecewise continuous  $u(t)$  having piecewise continuous first and second derivatives will be called an *admissible control*. The constraints on  $u$  may depend on  $t$  and  $x$ , and are expressed by

$$R(t, x, u) \leq 0, \quad R = (R^1, \dots, R^r) \quad (2)$$

where the functions  $R^i$ ,  $i = 1, \dots, r$  satisfy the *constraint conditions*:

- (i) If  $r > m$ , then at each  $(t, x, u)$  at most  $m$  components of  $R$  can vanish.
- (ii) At each  $(t, x, u)$  the matrix  $(\partial R^i / \partial u^j)$ , where  $i$  ranges over those indices where  $R^i(t, x, u) = 0$  and  $j = 1, \dots, m$ , has maximum rank.

These conditions preclude state space constraints. An admissible control satisfying the constraint (2) will be called *permissible*. The objective of control is to minimize, over the admissible controls, the cost functional

$$J(u) = \int_{t_0}^{t_1} f^0(t, x, u) dt, \quad (3)$$

subject to (1), (2), and some terminal conditions on  $x(t_1)$ . For simplicity, the terminal state and time will be fixed given values

$$t = t_1 \text{ is fixed, } \quad x(t_1) = x_1. \quad (4)$$

The functions  $f^0, f$ , and  $R$  are assumed to be twice continuously differentiable in all arguments.

To summarize, the *primal optimal control problem P* is

(P) *Minimize over the admissible controls the cost functional (3) subject to the differential equation (1), the constraint (2), and the terminal condition (4).*

The necessary conditions for this problem are well known [6]. As usual, we define the pre-Hamiltonian function  $H$  by

$$H(t, x, u, p) = p^0 f^0(t, x, u) + p f(t, x, u). \quad (5)$$

If an admissible control  $u^*(t)$ ,  $t_0 \leq t \leq t_1$ , is optimal and  $x^*(t)$ ,  $t_0 \leq t \leq t_1$ , is the corresponding trajectory [solution of (1)], then there exists a constant  $p^0 \geq 0$ , an  $n$ -vector  $p(t) = p^*(t)$  continuous on  $[t_0, t_1]$  such that  $(p^0, p^*(t)) \neq 0$  and an  $r$ -vector  $\mu(t) = \mu^*(t) \geq 0$  continuous on  $[t_0, t_1]$  except perhaps at corners of  $x^*(t)$  where it possesses unique right and left-hand limits, such that the following conditions hold.

The Euler equations:

$$\dot{x} = H_p \quad (6)$$

$$\dot{p} = -(H_x + \mu R_x), \quad (7)$$

$$H_u + \mu R_u = 0, \quad (8)$$

$$\mu^i R^i = 0, \quad i = 1, \dots, r, \quad \mu \geq 0. \quad (9)$$

The Weierstrass-Pontryagin condition:

For all permissible  $u$  (i.e., satisfying (2)) and for all  $t \in [t_0, t_1]$ ,

$$H(t, x^*, u^*, p^*) \leq H(t, x^*, u, p^*). \quad (10)$$

It will be assumed henceforth that the optimal trajectory is normal, i.e.,  $p^0 \neq 0$  and can be chosen as  $p^0 = 1$ .

Denote by

$$u = c(t, x, p) \quad (11)$$

the value of a permissible  $u$  which minimizes  $H(t, x, u, p)$  at a point  $(t, x, p)$  where such a minimum exists, and let

$$H(t, x, c(t, x, p), p) = \mathcal{H}(t, x, p). \quad (12)$$

If  $c(t, x, p)$  is continuously differentiable in  $(t, x, p)$  and single valued then  $\mathcal{H}$  is the Hamiltonian function; namely,

$$\dot{x} = H_p|_{u=c} = \mathcal{H}_p \quad (13)$$

and

$$\dot{p} = -(H_x + \mu R_x)|_{u=c} = -\mathcal{H}_x. \quad (14)$$

To show this, we note that

$$\mathcal{H}_p = H_p + H_u c_p, \quad (15)$$

$$\mathcal{H}_x = H_x + H_u c_x, \quad (16)$$

and that the minimizing  $u$  in (11) must satisfy (8) and (9), because (8) and (9) are necessary for (10). Thus, differentiating  $\mu R(t, x, c(t, x, p)) \equiv 0$  with respect to  $x$  and to  $p$  gives

$$\mu R_x + \mu R_u c_x = 0 \quad \text{and} \quad \mu R_u c_p = 0,$$

respectively. Multiplying (8) by the matrices  $c_x$  and  $c_p$  and using the above gives

$$H_u c_p = 0 \quad \text{and} \quad H_u c_x = \mu R_x$$

respectively, which converts (15) and (16) into (13) and (14), respectively (see also [7] for a similar result).

### 3. CONVEXITY ASSUMPTIONS

The following assumptions on the primal problem are made.

(A<sub>1</sub>) *The function  $f^0(t, x, u) + \mu R(t, x, u)$  is strictly convex in  $x$  and  $u$  for all  $t \in [t_0, t_1]$  and all  $\mu \geq 0$ .*

Let  $f^* = f(t, x^*, u^*)$ , etc. Assumption A<sub>1</sub> implies that

$$f^{0*} + \mu R^* \geq f^0 + \mu R + (f_x^0 + \mu R_x)(x^* - x) + (f_u^0 + \mu R_u)(u^* - u), \quad (17)$$

with equality if and only if  $x = x^*$  and  $u = u^*$ .

(A<sub>2</sub>) *There exist open regions  $N(x^*)$  and  $N(p^*)$  in  $R^n$  and an open region  $N(u^*)$  in  $R^m$  containing all the values of the optimal  $x^*(t)$ ,  $p^*(t)$  and  $u^*(t)$ ,  $t_0 \leq t \leq t_1$ , respectively, and such that for all  $x(t)$ ,  $p(t)$ , and  $u(t)$  satisfying (1),*

(7), (8), and (9), respectively, and lying entirely in the respective regions, the functional

$$\int_{t_0}^{t_1} pf(t, x, u) dt \quad \text{is convex in } x \text{ and } u \text{ for all } p(t) \in N(p^*).$$

This implies that

$$\int_{t_0}^{t_1} pf^* dt \geq \int_{t_0}^{t_1} [pf + (pf_x)(x^* - x) + (pf_u)(u^* - u)] dt. \quad (18)$$

For the reciprocal problem  $R_5$  (next section) we assume in addition that

(A<sub>3</sub>) The function  $c(t, x, p)$  of (11) is uniquely defined for all  $t \in [t_0, t_1]$ ,  $x \in N(x^*)$ ,  $p \in N(p^*)$ , except possibly at isolated points  $(t, x, p)$ .

Since (8) and (9) are necessary for the minimum of  $H(t, x, p, u)$  subject to (2), a  $\mu \geq 0$  such that  $u = c(t, x, p)$  of A<sub>3</sub> satisfies (8) and (9) can be found.

For  $R_6$  it will be assumed that, in addition to A<sub>1</sub>, A<sub>2</sub> and A<sub>3</sub>,

(A<sub>4</sub>) The function  $c(t, x, p)$  in A<sub>3</sub> is continuously differentiable in  $(t, x, p)$ , and further, the Hamiltonian  $\mathcal{H}(t, x, p)$  in (12) is strictly convex on  $N(x^*)$  for all  $p \in N(p^*)$  and for all  $t \in [t_0, t_1]$ , and is twice differentiable in  $x$ .

The strict convexity of  $\mathcal{H}$  implies

$$\mathcal{H}(t, x^*, p) \geq \mathcal{H}(t, x, p) + (x^* - x) \mathcal{H}_x(t, x, p) \quad (19)$$

with equality if and only if  $x = x^*$ . Since  $\mathcal{H}$  is twice differentiable, it also implies that the matrix  $\mathcal{H}_{xx}$  is positive definite for all  $x \in N(x^*)$ ,  $p \in N(p^*)$  and  $t \in [t_0, t_1]$ .

By assumption A<sub>4</sub>, but not by A<sub>3</sub>,  $u(t) = c(x(t), p(t), t)$  is continuous in  $t$ , thus precluding the important case of discontinuous controls (switching) for the reciprocal problem  $R_6$ . The convexity assumption in A<sub>1</sub> is frequently fulfilled in practice. The convexity assumptions in A<sub>2</sub> and A<sub>4</sub> may seem contrived; however, in Section 6 they will be shown to imply the convexity of the "surface of minimum cost" which is not unreasonable. In particular, for linear systems, A<sub>2</sub> holds trivially for all  $x, u$ , and  $p$ , because

$$f(t, x, u) = A(t)x + B(t)u = f_x x + f_u u,$$

and (18) is satisfied with an equality for all  $x, u$ , and  $p$ .

#### 4. THE RECIPROCAL PROBLEMS

The following problems will be shown in the next section to be reciprocal to the primal problem P. Here, the vector valued function  $\mu(t)$  will be considered admissible if it is piecewise continuous.

(R<sub>1</sub>) Maximize, over admissible  $u$  with values in  $N(u^*)$  and admissible  $\mu$ , the functional

$$I_1(u, \mu) = \int_{t_0}^{t_1} [f^0(t, x, u) + \mu R(t, x, u)] dt + x(t_1)p(t_1) - x_1p(t_1) \quad (20)$$

subject to

$$\dot{x} = f(t, x, u), \quad x(t_0) = x_0 \quad (1)$$

$$\dot{p} = -(H_x + \mu R_x), \quad (21)$$

$$H_u + \mu R_u = 0, \quad \mu \geq 0. \quad (22)$$

By substituting

$$x(t_1)p(t_1) = \int_{t_0}^{t_1} (\dot{x}p + x\dot{p}) dt + x_0p(t_0) \quad (23)$$

and using (1) and (21) we get the equivalent reciprocal problem

(R<sub>2</sub>) Maximize, over admissible  $u$  with values in  $N(u^*)$  and admissible  $\mu$ , the functional

$$I_2(u, \mu) = \int_{t_0}^{t_1} [f^0 + \mu R - x(H_x + \mu R_x) + pH_p] dt - x_1p(t_1) + x_0p(t_0)$$

subject to (1), (21) and (22) (as in R<sub>1</sub>).

By adding the constraint  $x(t_1) = x_1$  to R<sub>1</sub>, the reciprocal problem, derived in [1] for linear systems, is obtained:

(R<sub>3</sub>) Maximize, over admissible  $u$  with values in  $N(u^*)$  and admissible  $\mu$  the functional

$$I_3(u, \mu) = \int_{t_0}^{t_1} [f^0 + \mu R] dt \quad (25)$$

subject to

$$\dot{x} = f(t, x, u), \quad x(t_0) = x_0, \quad x(t_1) = x_1,$$

$$\dot{p} = -(H_x + \mu R_x),$$

$$H_u + \mu R_u = 0, \quad \mu \geq 0.$$

By adding the constraint  $\mu^i R^i = 0, i = 1, \dots, r$  to R<sub>2</sub> we get

(R<sub>4</sub>) Maximize over the admissible  $u$  with values in  $N(u^*)$  and admissible  $\mu$  the functional

$$I_4(u, \mu) = \int_{t_0}^{t_1} [f^0 - xH_x + pH_p] dt - x_1p(t_1) + x_0p(t_0) \quad (26)$$

subject to

$$\begin{aligned} \dot{x} &= f(t, x, u), & x(t_0) &= x_0, \\ \dot{p} &= -(H_x + \mu R_x), \\ H_u + \mu R_u &= 0, & \mu &\geq 0, & \mu^i R^i &= 0, & i &= 1, \dots, r. \end{aligned}$$

By assuming A<sub>3</sub>, and substituting (11) for  $u$ , the problem R<sub>4</sub> converts into

(R<sub>5</sub>) Maximize, over functions  $x(t)$  with values in  $N(x^*)$ , the functional

$$I_5(x) = \int_{t_0}^{t_1} [f^0 - xH_x + pH_p] \Big|_{u=c(t, x, p)} dt - x_1p(t_1) + x_0p(t_0) \quad (27)$$

subject to

$$\begin{aligned} \dot{x} &= f(t, x, c(t, x, p)), & x(t_0) &= x_0, \\ \dot{p} &= -(H_x + \mu R_x) \Big|_{u=c(t, x, p)} \end{aligned} \quad (28)$$

It will be shown that by assumption A<sub>4</sub> the differential constraint (28) in R<sub>5</sub> can be removed. In addition, A<sub>4</sub> implies (13) and (14). Thus we have

(R<sub>6</sub>) Maximize, over  $x(t)$  with values in  $N(x^*)$  the functional

$$I_6(x) = \int_{t_0}^{t_1} [f^0(t, x, c) - x\mathcal{H}_x + p\mathcal{H}_p] dt - x_1p(t_1) + x_0p(t_0),$$

subject to

$$\dot{p} = -\mathcal{H}_x(t, x, p)$$

The derivation of these reciprocal problems is summarized in Fig. 1.

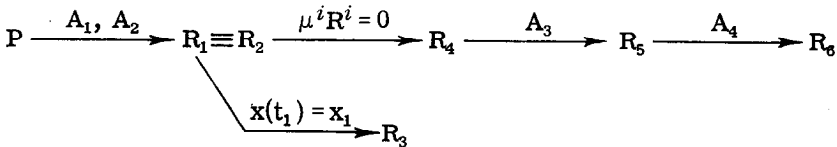


FIG. 1. A chart depicting the evolution of the reciprocal problems. The items above the arrow are the additional assumptions and constraints used to derive the reciprocal problem.

## 5. RECIPROCITY THEOREM

THEOREM. *If the functions  $u^*(t)$ ,  $x^*(t)$ ,  $p^*(t)$ ,  $\mu^*(t)$ ,  $t_0 \leq t \leq t_1$ , minimize the primal problem P and the assumptions  $A_1$  and  $A_2$  hold, then these functions maximize the reciprocal problems  $R_1$ ,  $R_2$ ,  $R_3$  and  $R_4$ , and*

$$\min J = \max I_1 = \max I_2 = \max I_3 = \max I_4 ; \quad (29)$$

furthermore, the functions maximizing any of  $R_1$ ,  $R_2$ ,  $R_3$  or  $R_4$ , also maximize each of the other reciprocal problems and minimize P. If in addition  $A_3$  holds then the above is true also for  $R_5$ ; if  $A_4$  is fulfilled too, then the above holds for  $R_6$  and moreover, the Hamiltonians of  $R_6$  and of P are identical.

Thus, the existence of the minimum of P implies the existence of the maxima for the reciprocal problems, but not conversely. If the optimizing functions of P are unique, so are those of the reciprocal problems.

PROOF. Consider  $R_1$  first. The minimizing functions  $u^*$ ,  $x^*$ ,  $p^*$ , and  $\mu^*$  satisfy the constraints (1), (2), and (4), and the necessary conditions (6) through (9). Thus, these functions satisfy all the constraints of  $R_1$ . Furthermore, by substitution,  $J^* = I_1^*$ .

It remains to show that  $I_1^* \geq I_1$ , for all  $u \in N(u^*)$ ,  $x \in N(x^*)$ , and  $p \in N(p^*)$ , satisfying the constraints of  $R_1$ .

$$J^* = I_1^* = \int_{t_0}^t f^{0*} dt \geq \int_{t_0}^{t_1} (f^{0*} + \mu R^*) dt$$

because  $\mu \geq 0$  and  $R^* \leq 0$ . By the convexity assumption  $A_1$ , using (17), we have

$$\geq \int_{t_0}^{t_1} [f^0 + \mu R + (f_x^0 + \mu R_x)(x^* - x) + (f_u^0 + \mu R_u)(u^* - u)] dt$$

and using the constraints (21) and (22)

$$= \int_{t_0}^{t_1} [f^0 + \mu R - (pf_x)(x^* - x) - (pf_u)(u^* - u) - \dot{p}(x^* - x)] dt.$$

Eliminating  $\dot{p}$  by integration by parts gives

$$\begin{aligned} &= \int_{t_0}^{t_1} [f^0 + \mu R + p(f^* - f) - (pf_x)(x^* - x) - (pf_u)(u^* - u)] dt \\ &\quad - (x^*(t_1) - x(t_1))p(t_1), \end{aligned}$$



since  $(x^*(t_0) - x(t_0)) = (x_0 - x_0) = 0$ . By the convexity assumption  $A_2$ , using (18), we have further

$$\geq \int_{t_0}^{t_1} [f^0 + \mu R] dt + x(t_1) p(t_1) - x_1 p(t_1) = I_1.$$

Thus,  $I_1^* \geq I_1$ .

Since  $R_2 \equiv R_1$ , and since  $R_3$  is derived from  $R_1$  and  $R_4$  and  $R_5$  are derived from  $R_2$  by adding constraints and assumptions, it follows that  $\min P \geq R_1 \geq R_3$  and  $R_1 = R_2 \geq R_4 \geq R_5$ . Here,  $P$ ,  $R_1$ , etc., is the value of the respective functional subject to the constraints. Further, since all these additional constraints are satisfied by  $u^*$ ,  $x^*$ ,  $p^*$ , and  $\mu^*$ , we have (29).

If the minimum of  $P$  exists, then  $J^* = I_1^*$  is as the above proof shows the maximum of  $R_1$ ,  $R_2$ ,  $R_3$ ,  $R_4$  and  $R_5$ , which by the strict convexity assumption  $A_1$  is achieved if and only if  $u = u^*$  and  $x = x^*$ .

To complete the proof it only remains to show that under assumption  $A_4$ ,  $R_5 \equiv R_6$ . The pre-Hamiltonian for  $R_6$  is

$$G(t, x, p, \lambda) = \lambda^0 [f^0(t, x, c) - x \mathcal{H}_x + p \mathcal{H}_p] - \lambda \mathcal{H}_x = \mathcal{H} - (x + \lambda) \mathcal{H}_x, \quad (30)$$

where by normality,  $\lambda^0 = 1$  (normality is assured in this case by the corollary to Theorem 3 in [6]). A necessary condition for the maximization of  $R_6$  is the maximization of  $G$  with respect to  $x \in N(x^*)$ . Since  $N(x^*)$  is open, this requires

$$G_x = \mathcal{H}_x - (x + \lambda) \mathcal{H}_{xx} - \mathcal{H}_x = - (x + \lambda) \mathcal{H}_{xx} = 0, \quad (31)$$

and since  $\mathcal{H}_{xx}$  is positive definite, (31) gives

$$x = -\lambda. \quad (32)$$

By differentiating (31) we find that

$$G_{xx} |_{x=-\lambda} = -\mathcal{H}_{xx}$$

is negative definite. Thus,  $x = -\lambda$  maximizes  $G$ . The Hamiltonian  $G^0$  for  $R_6$  is, from (30) with (32),

$$G |_{x=-\lambda} = G^0 = \mathcal{H} \quad (33)$$

as claimed in the theorem. The costate  $\lambda$  for  $R_6$  must satisfy

$$\dot{\lambda} = -G_p^0 = -\mathcal{H}_p$$

and thus in view of (32)  $x(t)$  maximizing  $R_6$  must satisfy

$$\dot{x} = \mathcal{H}_p = f(t, x, c(t, x, p)).$$

Further, the transversality condition requires that

$$-x_1 dp(t_1) + x_0 dp(t_0) = \lambda(t_1) dp(t_1) - \lambda(t_0) dp(t_0)$$

for all  $dp(t_1)$  and all  $dp(t_0)$ . This implies

$$\lambda(t_0) = -x_0 \quad \text{and} \quad \lambda(t_1) = -x_1.$$

Hence, the  $x(t)$  maximizing  $R_6$  must satisfy the differential constraint (28) of  $R_5$ . *Q.E.D.*

## 6. A GEOMETRIC INTERPRETATION OF THE CONVEXITY ASSUMPTIONS

Consider trajectories  $x(t)$  which are solutions of

$$\dot{x} = f(t, x, c(t, x, p)), \quad x(t_0) = x_0, \quad (34)$$

$$\dot{p} = -(H_x + \mu R_x)|_{u=c(t, x, p)}, \quad (35)$$

and have end points  $x(t_1)$  in a neighborhood  $N(x_1) \subset N(x^*)$  of  $x_1$ . It is assumed that

(1) At each point  $x(t_1)$  in  $N(x_1)$  the functional  $J(u)$  in (3) has a minimal value with respect to permissible controls, which will be denoted by  $S(x(t_1))$ . Clearly,

$$S(x(t_1)) = \int_{t_0}^{t_1} f^0(t, x, c(t, x, p)) dt \quad (36)$$

subject to (34) and (35).

(2) The function  $S(x(t_1))$  is smooth and strictly convex on  $N(x_1)$ . This is not an unreasonable assumption to make. Thus

$$S(x_1) \geq S(x(t_1)) + (x_1 - x(t_1)) S_{x(t_1)} \quad (37)$$

with equality if and only if  $x(t_1) = x_1$ .

The function  $S(x(t_1))$  describes, on  $N(x_1)$ , a surface of minimum "cost" of transferring the state from the fixed point  $(x_0, t_0)$  to  $(x, t_1)$ , and it is well known that

$$p(t_1) = -S_{x(t_1)}. \quad (38)$$

Thus, (37) can be written as

$$S(x_1) = \min J = \max_{x(t_1) \in N(x_1)} [S(x(t_1)) + (x(t_1) - x_1) p(t_1)]. \quad (39)$$

Substituting (23) and (36) in (39) we have

$$\min J = \max_{x \in N_x} \left[ \int_{t_0}^{t_1} (f^0 - xH_x + pH_p) \Big|_{u=c} dt - x_1 p(t_1) + x_0 p(t_0) \right] \quad (40)$$

subject to (34) and (35).

Relation (40) is the reciprocal problem  $R_5$ . The equivalence of (40) with (37), under the assumption that  $S(x(t_1))$  is smooth and thus allowing (38), shows that the convexity assumptions in  $A_1$  and  $A_2$  imply the convexity of  $S(x(t_1))$ . Thus, the convexity assumption  $A_2$  can be expected to hold in some cases of interest.

Under the assumptions made here, the primal problem P is solved by solving (34) and (35) with  $x(t_1) = x_1$ . Thus, the maximization in (39) or (40) simply replaces the terminal condition  $x(t_1) = x_1$ , and under the strict convexity of  $S(x(t_1))$  there is a maximum if and only if the terminal condition is satisfied. If  $x(t_1) \neq x_1$  then (39) provides a lower bound on  $\min J$ .

The convexity assumption in  $A_4$  implies the convexity of  $S(x(t_1))$  also. This is easily established by integrating (19) between  $t_0$  and  $t_1$  along trajectories and observing that this leads to (37). In fact,  $A_4$  implies considerably more. Let

$$S(x(t)) = \int_{t_0}^t f^0(t, x, c(t, x, p)) dt,$$

denote the value of the above integral when integrated along solutions of (34) and (35) lying in  $N(x^*)$ , and let  $N(x(t)) \subset N(x^*)$  be a region containing the end points  $x(t)$  of these solutions. Then, integrating (19) between  $t_0$  and  $t \in [t_0, t_1]$  shows that  $S(x(t))$  is strictly convex on  $N(x(t))$  for all  $t \in [t_0, t_1]$ .

## 7. CONCLUSION

Of the reciprocal problems derived here,  $R_6$  has the least number of constraints and is the most similar to the primal problem P. However, it requires the strongest convexity assumption, assumption  $A_4$ . It is also in a sense antisymmetrical to P in that there are no end conditions at all on the differential constraint of  $R_6$  while that of P has the end conditions  $x(t_0) = x_0$  and  $x(t_1) = x_1$ . If in P some of the components of  $x(t_1)$  are open, then the corresponding components of  $p(t_1)$  in  $R_6$  will be fixed by the natural boundary condition  $p^i(t_1) = 0$ .

The problems  $R_5$  and  $R_6$  have the advantage over the other reciprocal problems in having no inequality constraints. On the other hand, there are  $n$  "control variables" in  $R_5$  and  $R_6$  while  $R_1$  through  $R_4$  are maximized with

respect to  $m + r$  variables ( $m$  controls  $u^i$  with no inequality constraint and  $r$  variables  $\mu^i$  constrained by  $\mu^i \geq 0$ ), which in many problems can be considerably less than  $n$ . In general, the choice of a reciprocal problem will depend on the type of iterative technique used to solve the primal problem; one would choose that reciprocal problem where the (reciprocal) constraints are as closely or as easily as possible satisfied by the functions resulting from the iterative solution. For example,  $R_5$  or  $R_6$  may be best suited when the primal problem is attacked by solving iteratively the two-point boundary problem

$$\dot{x} = f(t, x, c(t, x, p)), \quad x(t_0) = x_0, \quad x(t_1) = x_1,$$

$$\dot{p} = -H_x|_{u=c(t, x, p)}.$$

A solution  $p(t)$  and  $x(t)$  with  $x(t_0) = x_0$  satisfies the constraints of  $R_5$  and thus substitution of  $p(t)$  and  $x(t)$  into  $I_5$  in (27) gives a lower bound on the cost functional  $J(u)$  of  $P$ .

The practical value of the reciprocity theorem as an aid in numerical solutions remains to be tested, in particular for non-linear systems. The main obstacle in the nonlinear case is assumption  $A_2$ , satisfied trivially for linear systems. However, as shown,  $A_2$  is not unreasonable. Assumption  $A_1$  can be readily checked from the data of  $P$  and is frequently applicable; unfortunately, it precludes formulating a reciprocal of the type considered here for the important case of the time-optimal problem (except when  $R(x, u, t)$  is strictly convex in  $x$  and  $u$ ).

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